ON THE BOUNDARY BEHAVIOR OF TEICHMUELLER GEODESICS (Complex Analysis and Topology of Discrete Groups and Hyperbolic Spaces)

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ON THE BOUNDARY BEHAVIOR OF TEICHMÜLLER GEODESICS

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ABSTRACT. We investigate accumulation points of Teichmüller geodesic rays in the Thurston compactification of a Teichmüller space. We show that the Thurston boundary does not consist only of accumulation points of rays. Moreover, we find a topological relation between the vertical foliation associated with a ray and the measured foliation representing an accumulation point of the ray.

1. BACKGROUND

The Teichmüller space $T(X)$ of a closed surface X of genus $g \geq 2$ can be defined as the space of conformal structures on the surface. Teichmüller geodesic rays are described in terms of quadratic differentials; any ray is given by contracting the horizontal foliation of a quadratic differential and by expanding the vertical one. We shall investigate the boundary behaviors of Teichmüller rays in a compactification of $T(X)$.

Alternatively, $T(X)$ can be viewed as the space of metrics of curvature -1 on the surface X. There is a natural compactification of $T(X)$, called Thurston's compactification due to Thurston in view of hyperbolic geometry. Using hyperbolic length functions of simple closed curves on X, Thurston [FLP] gave a compactification in $T(X)$, which is independence of the base surface X. The action of the mapping class group extends to this boundary. Moreover, the boundary, called Thurston's boundary, can be viewed as the space $\mathcal{PMF}$ of all projective classes of measured foliations.

They are natural from one of these points of view: Tecihmüller geodesics from the point of view of conformal geometry and Thurston's compactification from the point of view of hyperbolic geometry. There is no obvious way to compare hyperbolic geometry and conformal geometry. It is of interest to formulate the boundary behavior of Teichmüller geodesics. In the Thurston compactification, Masur [Ma] showed that almost every Teichmüller ray converges to the vertical foliation associated with the quadratic differential. He also showed that infinitely many rays converge to a boundary point representing a rational measured foliation, which has only closed leaves.

Theorem A (Masur, [Ma]). *If φ is a Jenkins-Strebel differential, that is, if the vertical foliation F has only compact leaves, then associated ray converges in Thurston's boundary to the barycenter of the leaves (the foliation with the same closed leaves all of whose cylinders have unit height), while if φ is uniquely ergodic and minimal , it converges to the projective class of F.*

It is worth pointing out that in the case of $F = \sum_{i=1}^{N} a_i \alpha_i$, where α_i's are simple closed curves and a_i's are non-negative numbers, the Teichmüller geodesic associated with F converges to the barycenter $\left[\sum_{i=1}^{N} \alpha_i\right]$, rather than to the projective class of F. The question of the description of the behavior of an arbitrary Teichmüller geodesic in Thurston's compactification is still open. Recently, Lenzhen [Le] showed the following:

Theorem B. *There exists a Teichmüller geodesic ray which does not converge in the Thurston compactification.*

Lenzehn gave examples of geodesic rays having at least 2 accumulation points in the boundary. Therefore it is natural to consider the limit set of a ray defined as the set of all accumulation points of the ray. We find boundary points to which no geodesic accumurates in Thurston's compactification (Theorem 4.1).

Theorem 1.1. *Let $[G]$ be a point of the Thurston boundary represented by a rational measured foliation G supported by at least two simple closed curves. If the annuli foliated by closed leaves of G have different heights, then no Teichmüller ray accumulates to $[G]$.*

The aim is to investigate the shapes of limit sets (Theorem 5.1):

Theorem 1.2. *Every accumulation point of a Teichmüller ray is expressed as a sum over the same minimal components as those in the minimal decomposition of the vertical foliation.*

2. Preliminaries

Let X be a closed Riemann surface of genus g at least 2. A *marked Riemann surface* (Y, f) is a pair of a Riemann surface Y and a quasiconformal mapping $f : X \to Y$. Two marked Riemann surfaces (Y_1, f_1) and (Y_2, f_2) are said to be *Teichmüller equivalent* if there exists a conformal mapping $h\colon Y_1 \to Y_2$ such that h is homotopic to $f_2 \circ f_1^{-1}$. The set $T(X)$ of all Teichmüller equivalence classes of marked Riemann surfaces is called the *Teichmüller space* of X. The *Teichmüller distance* is defined to be

$$d_T([Y_1, f_1], [Y_2, f_2]) := \log \inf_h K(h),$$

where the infimum is taken over all qusiconformal mappings $h\colon Y_1 \to Y_2$ homotopic to $f_2 \circ f_1^{-1}$ and the maximal dilatation of h is denoted by $K(h)$. This gives $T(X)$ a complete distance function; the metric space $(T(X), d_T)$ is homeomorphic to the open ball $\mathbb{B}^{6g-6}$.

Recall that $T(X)$ is identified with the space of all equivalence classes of hyperbolic metrics on X with constant curvature -1. *The Thurston compactification* of $T(X)$ is the closure of the *Thurston embedding*

$$T(X) \ni \rho \mapsto [\alpha \mapsto \text{length}_\rho(\alpha)] \in \mathcal{PR},$$

where $\mathcal{PR} = ((\mathbb{R}_{\geq 0})^{\mathcal{S}} - \{0\})/\mathbb{R}_+$. The boundary of image of $T(X)$ is called *the Thurston boundary* of $T(X)$. Thurston showed that the boundary coincides with the set $\mathcal{PMF}$ of all projective measured foliations. He also showed that the boundary is homeomorphic to the sphere $\mathbb{S}^{6g-7}$ and the closure of image of $T(X)$ is homeomorphic to the closed ball $\mathbb{B}^{6g-6} \cup \mathbb{S}^{6g-7}$.

For any $t \geq 0$ and any holomorphic quadratic differential φ on X, there exists a unique Riemann surface X_t and a unique quadratic differential φ_t on X_t such that

$$w_t = e^{t/2}u + \sqrt{-1}e^{-t/2}v$$

are natural coordinates for φ_t away from zeros for all natural coordinates $w = u + \sqrt{-1}v$ of the quadratic differential φ. Suppose that $\mathcal{G}_t$ denotes the hyperbolic metric that uniformizes the Riemann surfaces X_t, then we obtain the path $t \mapsto \mathcal{G}_t$ from X on $T(X)$, which is called the *Teichmüller ray* associated with the vertical foliation of φ. It is known that every Teichmüller ray is a geodesic ray on $T(X)$, conversely, that every ray starting from X is given by the above construction.

A *pants decomposition* of a surface M of genus g at least 2 is a set of disjoint simple closed curves $\{\alpha_1, \ldots, \alpha_k\}$ $(k = 3g - 3)$ which decompose the surface M into pairs of pants. Given a pants decomposition

$\{\alpha_1, \ldots, \alpha_k\}$ on X, the *Fenchel-Nielsen coordinates* give a global parameterization of $T(X)$. Given a hyperbolic structure in $T(X)$, these coordinates consists of the length l_{α_i} of the geodesic representative of α_i, and the *twist parameters* t_{α_i}. The length l_{α_i} determine uniquely the geometry on each pair of pants, while the twist parameters are real numbers determining the way these pairs of pants are glued together to build up the hyperbolic surface.

For a metric σ on a surface M and a closed curve α on M, the *σ-length* of α, denoted by $\sigma(\alpha)$, is defined to be the infimum of the lengths in the homotopy class of α. Given a conformal structure X on M and a simple closed curve α on M, the *extremal length* $\mathrm{Ext}_X(\alpha)$ of α on X is defined to be the analytically quantity

$$\sup_{\sigma} \frac{\sigma(\alpha)^2}{\mathrm{area}_\sigma(M)},$$

where the supremum is taken over all conformal metric σ on X. It is well-known that the length coincides with the geometric quantity

$$\inf_{A} \frac{1}{\mathrm{Mod}(A)},$$

where $\mathrm{Mod}(A)$ denotes the modulus of an annulus A embedded into X with core homotopic to α, and the infimum is taken over such annuli. The following Lemma is well-known as Maskit's inequality.

Lemma 2.1 (Maskit [M]). *Let X be a conformal structure on a hyperbolic surface M, and let ρ denote the hyperbolic metric that uniformizes the Riemann surface X. Then the inequality*

$$2\exp\left(-\rho(\alpha)/2\right) \leq \frac{\rho(\alpha)}{\mathrm{Ext}_X(\alpha)} \leq \pi$$

holds for all simple closed curves α on M.

Extremal length and hyperbolic length are conformal invariants:

Lemma 2.2 (Wolpert [Wo]). *Let X_1, X_2 be conformal structures on a hyperbolic surface M, and let ρ_1, ρ_2 denote the hyperbolic metrics that uniformize the Riemann surfaces X_1, X_2, respectively. Then the inequalities*

$$e^{-d_T(X_1,X_2)} \leq \frac{\mathrm{Ext}_{X_2}(\alpha)}{\mathrm{Ext}_{X_1}(\alpha)} \leq e^{d_T(X_1,X_2)}, \quad e^{-d_T(X_1,X_2)} \leq \frac{\rho_2(\alpha)}{\rho_1(\alpha)} \leq e^{d_T(X_1,X_2)}$$

hold for all simple closed curves α on M.

3. Lengths and Twists along Geodesic Rays

Here and subsequently, the notation $\mathcal{G}_{F,X} = \{\mathcal{G}_t\}_{t\geq 0}$ denotes the Teichmüller ray determined by a measured foliation F on a Riemann surface X of genus $g \geq 2$.

To simplify our presentation, we will use the notation $\asymp$, $\prec$, $\succ$: for two functions f, g, the symbol $f \prec g$ means that the inequality $f \leq cg$ holds for some constant $c > 0$ independent of the parameter t. Equivalently, $f \succ g$ means that $f \geq g/c$, and $f \asymp g$ means that both $f \prec g$ and $f \succ g$ hold.

We need the following estimates.

Lemma 3.1. *The following holds for all $\alpha \in \mathcal{S}$.*

(1) *If $i(F,\alpha) \neq 0$, then the inequality*

$$t + 2\log i(F,\alpha) \leq \mathcal{G}_t(\alpha) \leq e^{t/2}\sqrt{2\pi|\chi(X)|\mathrm{Ext}_X(\alpha)}$$

holds, and hence $t \prec \mathcal{G}_t(\alpha) \prec e^{t/2}$.

(2) *If $i(F,\alpha) = 0$, then the inequality*

$$\mathcal{G}_t(\alpha) \geq e^{-t} \inf_{\beta\in\mathcal{S}} \mathcal{G}_0(\beta)$$

holds, and hence $e^{-t} \prec \mathcal{G}_t(\alpha)$. Moreover if α is homotopic to the core curve of a maximal annulus for F, which is foliated by all closed leaves homotopic each other, then the inequality

$$\mathcal{G}_t \leq e^{-t}\pi/\mathrm{Mod}_X(\alpha),$$

holds, and hence $\mathcal{G}_t(\alpha) \asymp e^{-t}$.

(3) *If the condition $i(F,\alpha) = 0$ holds and if there is no maximal annuli for F with core homotopic to α, then the inequality $1/t \prec \mathcal{G}_t(\alpha)$ holds.*

The following lemma is due to Minsky.

Lemma 3.2 (Minsky, Lemma 8.3 in [Mi]). *For any $\alpha \in \mathcal{S}$ with the condition $i(F,\alpha) = 0$, the set $\{\mathcal{G}_t(\alpha)\}_{t\geq 0}$ is bounded above. Moreover, if α is a closed leaf (possibly singular leaf) of F, then $\mathcal{G}_t(\alpha)$ converges to 0 as t tends to ∞.*

Let $\{\alpha, \gamma_1, \dots, \gamma_{k-1}\}$ be a pants decomposition of X and let $\bar{\alpha}$ be a *dual curve* to α, that is, a curve intersecting α_i either once or twice and disjoint from γ_j for all $1 \leq j \leq k-1$. If $i(\alpha,\bar{\alpha}) = 1$, then α is on the boundary of just one pair of pants P (two boundary components of P are glued together along α). We denote the other boundary component of P by ω. If $i(\alpha,\bar{\alpha}) = 2$, the α is on the boundary of two different pants P, P'; let ω_1,ω_2,ω_1',ω_2' the other boundary components of P, P', respectively.

Applying Lemma 6.4 in [DS] to our case, we then have the following estimate:

Proposition 3.3. *With the above notation, suppose that the curve α is homotopic to a closed leaf of F.*

(1) *If $i(\alpha, \bar{\alpha}) = 1$, then the $\mathcal{G}_t$-length of $\bar{\alpha}$ is equal to*

$$2\log\frac{1}{\mathcal{G}_t(\alpha)} + \frac{1}{2}\mathcal{G}_t(\omega) + O(1).$$

(2) *If $i(\alpha, \bar{\alpha}) = 2$, then the $\mathcal{G}_t$-length of $\bar{\alpha}$ is equal to*

$$4\log\frac{1}{\mathcal{G}_t(\alpha)} + \max\{\mathcal{G}_t(\omega_1), \mathcal{G}_t(\omega_2)\} + \max\{\mathcal{G}_t(\omega_1'), \mathcal{G}_t(\omega_2')\} + O(1).$$

4. The shapes of limit sets

We say a measured foliation F is *rational* if it has only closed leaves. In this case, the closed leaves fall into homotopy classes $\alpha_1, \dots, \alpha_N$ and the set of homotopic leaves form an annulus, we write $F = \sum_{i=1}^N a_i\alpha_i$ in the setting that $a_i > 0$ is the height (with respect to the quadratic differential on X corresponding to F) of the annuli with core homotopic to α_i.

Then the following holds.

Theorem 4.1. *Let $[G]$ be a point of Thurston's boundary represented by a rational measured foliation G, denoted by $G = \sum_{i=1}^N b_i\alpha_i$. Then the following holds.*

(1) *If $b_i \neq b_j$ for some $i \neq j$, then there is no Tecihmüller ray such that the limit set contains $[G]$.*
(2) *If $b_1 = \cdots = b_N$, then the following three conditions are equivalent for any measured foliation F.*
   (a) $[G] \in L(\mathcal{G}_{F,X})$.
   (b) $F = \sum_{i=1}^N a_i\alpha_i$ *for some* $a_i > 0$.
   (c) $L(\mathcal{G}_{F,X}) = \{[\sum_{i=1}^N \alpha_i]\}$.

We say a foliation is *minimal* if it has only dense leaves after collapsing saddle connections. Let Σ denote the union of compact leaves of F joining singularities. It is well-known that each component of $X \setminus \Sigma$ is either an annulus swept out by closed leaves or a *minimal domain* in which every leaf is dense. Let Σ_C denote the union of noncontractible components of Σ. The boundary components of regular neighborhoods of Σ_C fall into homotopy classes $\{\alpha_1, \dots, \alpha_n\}$ which are represented by disjoint non-trivial circles. We then have the *minimal decomposition*

$$F = \sum_{\Omega} F_{\Omega} + \sum_{i=1}^{n} a_i \alpha_i$$

for some $a_i \geq 0$ and some minimal foliation F_Ω on a minimal domain Ω. We assume that $a_i = 0$ if the corresponding curve α_i is represented by a boundary component of a minimal domain, otherwise a_i indicates the height of the annular component with core homotopic to α_i.

Given a subsurface $Y \subset X$ whose boundary ∂Y consists of nontrivial circles, we will denote by $\mathcal{MF}_Y$ the space of Whitehead equivalence classes of measured foliations on Y. Recall that this space includes equivalence classes of only those foliations for which each component of the boundary ∂Y is a cycle and contains at least one singularity. Then the following proposition holds:

Proposition 4.2. *Suppose that a sequence ρ_n in $T(X)$ satisfies the condition that $\{\rho_n(\alpha_i)\}_n$ is bounded above for each $1 \leq i \leq N$ and that ρ_n converges to a foliation $[G]$ in Thurston's compactification. Then the representation G is written as the sum, may not be the minimal decomposition, of the form*

$$\sum_{\Omega} G_{\Omega} + \sum_{i=1}^{N} b_i \alpha_i,$$

where $b_i \geq 0$ and $G_\Omega \in \mathcal{MF}_\Omega \cup \{0\}$ is topological equivalent to F_Ω, so G_Ω is also minimal, unless $G_\Omega = 0$.

Proof. We see $i(G, \alpha_i) = 0$ for all i. This implies that the measured foliation G is written as the same sum for F, that is,

$$G = \sum_{\Omega} G_{\Omega} + \sum_{i=1}^{N} b_i \alpha_i$$

for some $b_i \geq 0$ and for some $G_\Omega \in \mathcal{MF}_\Omega \cup \{0\}$. Since $i(F_\Omega, G_\Omega) \leq i(F, G) = 0$, G_Ω either is topologically equivalent to F_Ω or 0. $\square$

5. Main Results

The following theorem is our main result. We give a topological description of accumulation points of rays:

Theorem 5.1. *Let F be a measured foliation with minimal decomposition of the form*

$$\sum_{\Omega} F_{\Omega} + \sum_{i=1}^{N} a_i \alpha_i,$$

and suppose that $\sum_\Omega F_\Omega \neq 0$. *If* $[G] \in L(\mathcal{G}_{F,X})$, *then* G *is written as the sum of the form*

$$\sum_\Omega G_\Omega + \sum_{i=1}^N b_i \alpha_i,$$

where $b_i \geq 0$ *and* $G_\Omega \in \mathcal{MF}_\Omega \cup \{0\}$ *satisfying the following properties.*

(1) $\sum_\Omega G_\Omega \neq 0$.
(2) F_Ω *and* G_Ω *are topologically equivalent unless* $G_\Omega = 0$.
(3) *If* $b_1 + \cdots + b_N > 0$, *then* $G_{\Omega'} \neq 0$ *for all minimal domains* Ω' *of* F.
(4) $a_i = 0$ *implies* $b_i = 0$.

Proof. By Lemma 3.2 and Proposition 4.2, we only need to show that the the properties (3), (4) hold, because the property (1) is an immediately consequence of Theorem 4.1. Let Ω_1 be a minimal domain such that $G_{\Omega_1} \neq 0$, and let $\beta_1 \in \mathcal{S}$ be a non-peripheral curve contained in Ω_1. Let $\{\alpha_1, \cdots, \alpha_N, \beta_1, \cdots, \beta_M\}$, possibly $M = 1$, be a pants decomposition of X, where $\alpha_1, \cdots, \alpha_N, \beta_1$ are as above, and let $\bar{\alpha}_i$ be a dual curve to α_i. We give the proof only for the case $i(\alpha_i, \bar{\alpha}_i) = 1$; the other case $i(\alpha_i, \bar{\alpha}_i) = 2$ is left to the reader.

Let us first show that the property (4) holds. By Proposition 3.3, we have

$$\mathcal{G}_t(\bar{\alpha}_i) = 2\log(1/\mathcal{G}_t(\alpha)) + 1/2\mathcal{G}_t(\omega) + O(1),$$

where ω is the boundary curve different from α of the pair of pants adjacent to α. The assumption $a_i = 0$ implies that α_i is represented by a boundary component of a minimal domain, so we have $\mathcal{G}_t(\alpha_i) \succ 1/t$ by Lemma 3.1 (3). Since $i(F, \beta_1) \neq 0$, we also have $\mathcal{G}_t(\beta_1) \geq t + O(1)$ by Lemma 3.1 (1), and hence

$$\frac{\log(1/\mathcal{G}_t(\alpha_i))}{\mathcal{G}_t(\beta_1)} \leq \frac{\log t + O(1)}{t + O(1)} \to 0.$$

Since there is a sequence $t_n \to \infty$ such that

$$\frac{\mathcal{G}_{t_n}(\omega)}{\mathcal{G}_{t_n}(\beta_1)} \to \frac{i(G,\omega)}{i(G,\beta_1)},$$

we get

$$\frac{\mathcal{G}_{t_n}(\bar{\alpha}_i)}{\mathcal{G}_{t_n}(\beta_1)} \to \frac{i(G,\omega)}{2i(G,\beta_1)}.$$

Hence $i(G,\omega) = 2i(G,\bar{\alpha}_i)$. It follows immediately that $b_i = 0$ if $i(G,\omega) = 0$; so we suppose that $i(G,\omega) \neq 0$, and then $\omega \in \{\beta_1, \ldots, \beta_M\}$. Since $i(\alpha_i, \bar{\alpha}_i) = 1$, there is just one minimal domain Ω_i of which α_i is one of boundary components. Then $i(G,\omega) = i(G_{\Omega_i}, \omega)$ and $i(G, \bar{\alpha}_i) = b_i + i(G_{\Omega_i}, \bar{\alpha}_i)$, and it suffices to show that $i(G_{\Omega_i}, \omega) \leq 2i(G_{\Omega_i}, \bar{\alpha}_i)$.

To see this, fix a orientation of the curves α_i, $\bar{\alpha}_i, \omega$ so that the concatenation $\alpha_i \cdot \bar{\alpha}_i \cdot (\alpha_i)^{-1} \cdot (\bar{\alpha}_i)^{-1}$ is freely homotopic to ω. We then have

$$i(G_{\Omega_i}, \omega) \leq 2i(G_{\Omega_i}, \alpha_i) + 2i(G_{\Omega_i}, \bar{\alpha}_i) = 2i(G_{\Omega_i}, \bar{\alpha}_i),$$

and (4) is proved.

Let us next show that the property (3) holds. Suppose $b_1 > 0$ for simplicity, then the property (4) gives $a_1 > 0$. This implies that $\omega \in \{\alpha_1, \ldots, \alpha_N\}$, so we have $\mathcal{G}_t(\omega) \prec 1$ by Lemma 3.2, and that $\mathcal{G}_t(\alpha_1) \asymp e^{-t}$ by Lemma 3.1 (2). Hence $\mathcal{G}_t(\bar{\alpha}_1) \asymp t$ by Proposition 3.3 (1). On the other hand, we have $\mathcal{G}_t(\alpha) \succ t$ for any non-peripheral curve $\alpha \in \mathcal{S}$ contained in Ω' by Lemma 3.1 (1). Hence $\mathcal{G}_t(\bar{\alpha}_1)/\mathcal{G}_t(\alpha) \prec 1$. It follows from the assumption $[G] \in L(\mathcal{G}_{F,X})$ that $i(G, \alpha) \neq 0$, this implies $G_{\Omega'} \neq 0$. □

The proof of the above theorem immediately gives the following corollary:

Corollary 5.2. *Under the same condition for Theorem 5.1, suppose that there exist a minimal domain Ω_0 with $G_{\Omega_0} \neq 0$ and a non-peripheral curve $\beta_0 \in \mathcal{S}$ contained in Ω_0 such that $\mathcal{G}_{t_n}(\beta_0)/t_n$ tends to ∞ if $\mathcal{G}_{t_n}$ converges to $[G]$. Then all b_i vanish.*

Now we consider the condition $(*)$ in terms of hyperbolic geometry for a minimal domain Ω_0 of F and for a subsequence $\mathcal{G}_{t_n}$ of $\{\mathcal{G}_t\}_{t \geq 0}$ that

> there exists a non-peripheral curve $\beta_0 \in \mathcal{S}$ contained in Ω_0 such that $\mathcal{G}_{t_n}$ is thick along β_0 for all n: that is,
>
> $$\inf_n \mathcal{G}_{t_n}(\beta_{t_n}) \neq 0,$$
>
> where $\beta_{t_n} \in \mathcal{S}$ denote the curve intersecting β_0 essentially with the shortest $\mathcal{G}_{t_n}$-length.

Proposition 5.3. *If the condition $(*)$ holds, then $\mathcal{G}_{t_n}(\beta_0) \asymp e^{t_n/2}$.*

Proof. By $(*)$, there is a thick component Y_{t_n} in which the $\mathcal{G}_{t_n}$-geodesic representative of β_0 is contained. It follows from Theorem 6 in [Ra2] that

$$\varphi_{t_n}(\beta_0) \asymp \lambda_{Y_{t_n}} \cdot \mathcal{G}_{t_n}(\beta_0),$$

where φ_{t_n} denotes the quadratic differential corresponding to $\mathcal{G}_{t_n}$ and $\lambda_{Y_{t_n}}$ the shortest φ_{t_n}-length over all non-peripheral, non-trivial, simple closed curves in Y_{t_n}. Since the diameter of Y_{t_n} with respect to the hyperbolic metric $\mathcal{G}_{t_n}$ is bounded above by a constant depending only on the topology of X, we have $\mathcal{G}_{t_n}(\beta_{t_n}) \prec 1$. It follows from Maskit's inequality and the analytic definition of extremal length that $\varphi_{t_n}(\beta_{t_n}) \prec$

1, hence we have $\lambda_{Y_{t_n}} \prec 1$. Since $\varphi_{t_n}(\beta_0) \asymp e^{t_n/2}$ because of $i(F, \beta_0) \neq 0$, we thus get $\mathcal{G}_{t_n}(\beta_0) \succ e^{t_n/2}$. On the other hand, Lemma 3.1 (1) gives $\mathcal{G}_{t_n}(\beta_0) \prec e^{t_n/2}$, hence $\mathcal{G}_{t_n}(\beta_0) \asymp e^{t_n/2}$. □

Consequently, we have the following:

Theorem 5.4. *Let F be a measured foliation with minimal decomposition of the form*

$$\sum_{\Omega} F_{\Omega} + \sum_{i=1}^{N} a_i \alpha_i,$$

and suppose that $\sum_{\Omega} F_{\Omega} \neq 0$. If $\mathcal{G}_{t_n}$ converges to $[G]$ and satisfies the condition $(*)$ for a minimal domain Ω_0, then G is written as the sum of the form

$$\sum_{\Omega} G_{\Omega}$$

where $G_{\Omega} \in \mathcal{MF}_{\Omega} \cup \{0\}$ is topologically equivalent to F_{Ω} unless $G_{\Omega} = 0$. Moreover $G_{\Omega_0} \neq 0$.

Proof. Theorem 5.1 gives

$$G = \sum_{\Omega} G_{\Omega} + \sum_{i=1}^{N} b_i \alpha_i,$$

where $b_i \geq 0$ and $G_{\Omega} \in \mathcal{MF}_{\Omega} \cup \{0\}$ satisfy the certain properties. If we prove $G_{\Omega_0} \neq 0$, it follows from Corollary 5.2 and Proposition 5.3 that $b_i = 0$. Since $G \neq 0$, there is $\alpha \in \mathcal{S}$ with $i(G, \alpha) \neq 0$, and hence

$$\frac{\mathcal{G}_{t_n}(\beta_0)}{\mathcal{G}_{t_n}(\alpha)} \to \frac{i(G, \beta_0)}{i(G, \alpha)}.$$

It follows from Lemma 3.1 that $\mathcal{G}_{t_n}(\alpha) \prec e^{t_n/2}$, and from Proposition 5.3 that $\mathcal{G}_{t_n}(\beta_0) \asymp e^{t_n/2}$. Hence we get $i(G_{\Omega_0}, \beta_0) = i(G, \beta_0) \neq 0$, this implies $G_{\Omega_0} \neq 0$.

□

We immediately obtain a sufficient condition for convergence of rays determined by foliations which have both minimal component and annular component:

Corollary 5.5. *Suppose that F is a measured foliation which has just one minimal domain Ω (note that F may have a annular component), and write it as*

$$F_{\Omega} + \sum_{i=1}^{N} a_i \alpha_i.$$

Suppose that any subsequence $\mathcal{G}_{t_n}$ of $\{\mathcal{G}_t\}_{t\geq 0}$ satisfies the property $(*)$. If F_Ω is uniquely ergodic on Ω, then $\mathcal{G}_t$ converges to $[F_\Omega]$.

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